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DEVELOPMENT OF EXPLICIT STIFFNESS AND MASS MATRICES FOR A TRIANGULAR PLATE ELEMENT

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Abstract—Explicit transverse bending stiffness and mass matrices are developed for a triangular finite element having a linear thickness variation. This element is more suitable for general application than the triangular and rectangular elements previously described for determining the static deflections and vibrational characteristics of plates in transverse bending.

The matrices are applied to determine the natural frequencies of vibration of a cantilever of square plan-form having a triangular cross-section.

INTRODUCTION

THE simplest finite elements for determining the transverse static deflections and vibrational characteristics of plates are the triangle and rectangle, the former being more suitable for components of non-rectangular or irregular plan-form. Stiffness and mass matrices for triangular elements of constant thickness have been developed by Severn and Taylor [1] based on a stress relationship over the area of the element, while Bazeley *et al.* [2] have defined a suitable displacement function. Matrices have also been developed by Dawe [3] for a rectangular element having a uniform thickness variation parallel to one pair of sides, but this element is limited in application. Argyris [4] has indicated a method for deriving the stiffness matrix in bending for a triangular element of non-uniform thickness although he does not appear to have produced the actual matrices in algebraic form.

Transverse stiffness and mass matrices are now developed in algebraic form for a general triangular element having a linear thickness variation so that the above approximation of average thickness is no longer necessary. The method of derivation is now well established and use is made of the deflection function described in [2] and by Zienkiewicz [5].

THE STIFFNESS MATRIX

The transverse stiffness of an arbitrary triangular plate element 123 of area Δ as shown in Fig. 1 is given by:

$$F = K\delta$$

$$\text{where} \quad F = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \delta = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix}$$

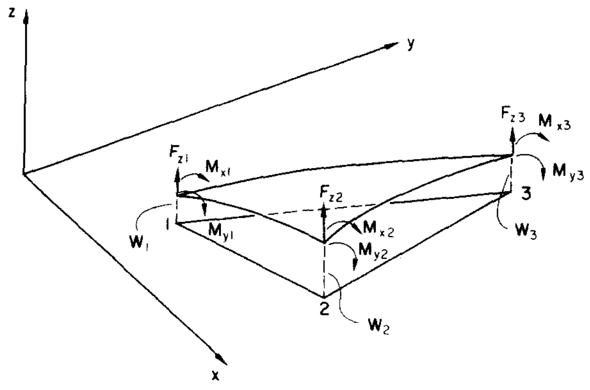

FIG. 1. Typical triangular element.

and K is the stiffness matrix of the element. F_1 and δ_1 represent the components of the forces and corresponding displacements acting at node 1 in the triangle as shown in the figure such that:

$$F_1 = \begin{Bmatrix} F_{z1} \\ M_{x1} \\ M_{y1} \end{Bmatrix} \quad \text{and} \quad \delta_1 = \begin{Bmatrix} W_1 \\ \theta_{x1} \\ \theta_{y1} \end{Bmatrix}.$$

By assuming a suitable displacement function, Zienkiewicz [5] has shown for an element of uniform thickness:

$$K = \int^y \int^x [B^*]^T[D][B^*]\,\mathrm{d}x\,\mathrm{d}y$$

where D is a 3×3 matrix relating the elastic properties of the material of the element and its thickness and B^* is a 3×6 matrix derived from the assumed displacement equation.

It is here assumed that the same displacement function applies but that the thickness of the element t varies linearly over the element, being t_1, t_2 and t_3 at the three nodes respectively. On substituting for the variable thickness and integrating over the area of the element it may be shown that the stiffness matrix reduces to

$$K = [F]^T[L][F]$$

where L is expressed in partitioned form as

$$L = \frac{E}{12(1-\nu^2)} \begin{bmatrix} R & \nu R & 0 \\ \nu R & R & 0 \\ 0 & 0 & 2(1-\nu)R \end{bmatrix}$$

Matrices F and R are given in Tables 1 and 2 respectively.

TABLE 1. MATRIX F

$\frac{1}{4\Delta^2}$

$-2(2b_1^2+b_2^2+b_3^2)$	$b_2b_3(b_3-b_2)$	$4b_1(b_2c_3-b_3c_2) +b_2b_3(c_3-c_2)$	$2b_2(b_2-2b_1)$	$-b_2b_3(2b_2+b_3-b_1)$	$b_2(b_3c_1-b_2c_3) -b_2c_3(b_2+b_3)$	$2b_3(b_3-2b_1)$	$b_2b_3(2b_3-b_1+b_2)$	$b_3c_2(b_2+b_3) +b_3(b_3c_2-b_2c_1)$
$2b_1(b_1-2b_2)$	$b_1b_3(2b_1-b_2+b_3)$	$b_1c_3(b_1+b_3) +b_1(b_1c_3-b_3c_2)$	$-2(b_1^2+2b_2^2+b_3^2)$	$b_1b_3(b_1-b_3)$	$4b_2(b_3c_1-b_1c_3) +b_1b_3(c_1-c_3)$	$2b_3(b_3-2b_2)$	$-b_1b_3(2b_3+b_1-b_2)$	$b_3(b_1c_2-b_3c_1) -b_3c_1(b_1+b_3)$
$2b_1(b_1-2b_3)$	$-b_1b_2(2b_1+b_2-b_3)$	$b_1(b_2c_3-b_1c_2) -b_1c_2(b_1+b_2)$	$2b_2(b_2-2b_3)$	$b_1b_2(2b_2-b_3+b_1)$	$b_2c_1(b_1+b_2) +b_2(b_2c_1-b_1c_3)$	$-2(b_1^2+b_2^2+2b_3^2)$	$b_1b_2(b_2-b_1)$	$4b_3(b_1c_2-b_2c_1) +b_1b_2(c_2-c_1)$
$-2(2c_1^2+c_2^2+c_3^2)$	$4c_1(b_3c_2-b_2c_3) +c_2c_3(b_3-b_2)$	$c_2c_3(c_3-c_2)$	$2c_2(c_2-2c_1)$	$c_2(b_1c_3-b_3c_2) -b_3c_2(c_2+c_3)$	$-c_2c_3(2c_2+c_3-c_1)$	$2c_3(c_3-2c_1)$	$b_2c_3(c_2+c_3) +c_3(b_2c_3-b_1c_2)$	$c_2c_3(2c_3-c_1+c_2)$
$2c_1(c_1-2c_2)$	$b_3c_1(c_1+c_3) +c_1(b_3c_1-b_2c_3)$	$c_1c_3(2c_1-c_2+c_3)$	$-2(c_1^2+2c_2^2+c_3^2)$	$4c_2(b_1c_3-b_3c_1) +c_1c_3(b_1-b_3)$	$c_1c_3(c_1-c_3)$	$2c_3(c_3-2c_2)$	$c_3(b_2c_1-b_1c_3) -b_1c_3(c_1+c_3)$	$-c_1c_3(2c_3+c_1-c_2)$
$2c_1(c_1-2c_3)$	$c_1(b_3c_2-b_2c_1) -b_2c_1(c_1+c_2)$	$c_1c_2(c_3-2c_1-c_2)$	$2c_2(c_2-2c_3)$	$b_1c_2(c_1+c_2) +c_2(b_1c_2-b_3c_1)$	$c_1c_2(2c_2-c_3+c_1)$	$-2(c_1^2+c_2^2+2c_3^2)$	$4c_3(b_2c_1-b_1c_2) +c_1c_2(b_2-b_1)$	$c_1c_2(c_2-c_1)$
$-2(2b_1c_1+b_2c_2+b_3c_3)$	$\frac{1}{2}b_3c_2(4b_1-b_2+b_3) -\frac{1}{2}b_2c_3(4b_1+b_2-b_3)$	$\frac{1}{2}b_2c_3(4c_1-c_2+c_3) -\frac{1}{2}b_3c_2(4c_1+c_2-c_3)$	$2(b_2c_2-b_1c_2-b_2c_1)$	$\frac{1}{2}b_2c_3(b_1-b_3) -\frac{1}{2}b_3c_2(4b_2+b_3-b_1)$	$\frac{1}{2}b_3c_2(c_1-c_3) -\frac{1}{2}b_2c_3(4c_2+c_3-c_1)$	$2(b_3c_3-b_1c_3-b_3c_1)$	$\frac{1}{2}b_2c_3(4b_3+b_2-b_1) +\frac{1}{2}b_3c_2(b_2-b_1)$	$\frac{1}{2}b_3c_2(4c_3+c_2-c_1) +\frac{1}{2}b_2c_3(c_2-c_1)$
$2(b_1c_1-b_2c_1-b_1c_2)$	$\frac{1}{2}b_3c_1(4b_1-b_2+b_3) +\frac{1}{2}b_1c_3(b_3-b_2)$	$\frac{1}{2}b_1c_3(4c_1+c_3-c_2) +\frac{1}{2}b_3c_1(c_3-c_2)$	$-2(b_1c_1+2b_2c_2+b_3c_3)$	$\frac{1}{2}b_1c_3(4b_2-b_3+b_1) -\frac{1}{2}b_3c_1(4b_2+b_3-b_1)$	$\frac{1}{2}b_3c_1(4c_2-c_3+c_1) -\frac{1}{2}b_1c_3(4c_2+c_3-c_1)$	$2(b_3c_3-b_2c_3-b_3c_2)$	$\frac{1}{2}b_3c_1(b_2-b_1) -\frac{1}{2}b_1c_3(4b_3+b_1-b_2)$	$\frac{1}{2}b_1c_3(c_2-c_1) -\frac{1}{2}b_3c_1(4c_3c_1-c_2)$
$2(b_1c_1-b_1c_3-b_3c_1)$	$\frac{1}{2}b_1c_2(b_3-b_2) -\frac{1}{2}b_2c_1(4b_1+b_2-b_3)$	$\frac{1}{2}b_2c_1(c_3-c_2) -\frac{1}{2}b_1c_2(4c_1+c_2-c_3)$	$2(b_2c_2-b_3c_2-b_2c_3)$	$\frac{1}{2}b_1c_2(4b_2+b_1-b_3) +\frac{1}{2}b_2c_1(b_1-b_3)$	$\frac{1}{2}b_2c_1(4c_2+c_1-c_3) +\frac{1}{2}b_1c_2(c_1-c_3)$	$-2(b_1c_1+b_2c_2+2b_3c_2)$	$\frac{1}{2}b_2c_1(4b_3-b_1+b_2) -\frac{1}{2}b_1c_2(4b_3+b_1-b_2)$	$\frac{1}{2}b_1c_2(4c_3-c_1+c_2) -\frac{1}{2}b_2c_1(4c_3+c_1-c_3)$

$b_1 = y_2-y_3 \qquad c_1 = x_3-x_2$
$b_2 = y_3-y_1 \qquad c_2 = x_1-x_3$
$b_3 = y_1-y_2 \qquad c_3 = x_2-x_1$

TABLE 2. MATRIX R

$20t_1^3$ $+2(t_2+t_3)(6t_1^2+t_2^2+t_3^2)$ $+6t_1(t_2^2+t_2t_3+t_3^2)$	$4t_1^3$ $+(2t_2+t_3)(3t_1^2+t_3^2)$ $+2t_1(3t_2^2+2t_2t_3+t_3^2)$ $+t_2^2(4t_2+3t_3)$	$4t_1^3$ $+(2t_3+t_2)(3t_1^2+t_2^2)$ $+2t_1(t_2^2+2t_2t_3+3t_3^2)$ $+t_3^2(3t_2+4t_3)$
$4t_1^3$ $+(2t_2+t_3)(3t_1^2+t_3^2)$ $+2t_1(3t_2^2+2t_2t_3+t_3^2)$ $+t_2^2(4t_2+3t_3)$	$20t_2^3$ $+2(t_1+t_3)(t_1^2+6t_2^2+t_3^2)$ $+6t_2(t_1^2+t_1t_3+t_3^2)$	$4t_3^3$ $+(t_1+2t_2)(t_1^2+3t_3^2)$ $+2t_3(t_1^2+2t_1t_2+3t_2^2)$ $+t_2^2(3t_1+4t_2)$
$4t_1^3$ $+(2t_3+t_2)(3t_1^2+t_2^2)$ $+2t_1(t_2^2+2t_2t_3+3t_3^2)$ $+t_3^2(3t_2+4t_3)$	$4t_3^3$ $+(t_1+2t_2)(t_1^2+3t_3^2)$ $+2t_3(t_1^2+2t_1t_2+3t_2^2)$ $+t_2^2(3t_1+4t_2)$	$20t_3^3$ $+2(t_1+t_2)(t_1^2+t_2^2+6t_3^2)$ $+6t_3(t_1^2+t_1t_2+t_2^2)$

THE MASS MATRIX

The mass matrix for the element may also be derived in the manner described by Zienkiewicz [5]. He indicated that it may be denoted by the expression:

$$M = \int [N]^T \rho [N] \, \mathrm{d(vol)}.$$

Column matrix N contains the shape functions defining the displacement of a point x, y in the element in terms of the nodal displacements while ρ is the density of the material of the plate. It is now assumed that the thickness t varies linearly within the triangular element and the same displacement function is employed. The mass matrix as given in Table 3 is then obtained.

VIBRATIONS OF TAPERED PLATES

The individual stiffness and mass matrices for each triangular element in the plate are combined with adjacent elements to obtain the overall stiffness matrix ($\bar{S}$) and mass matrix ($\bar{M}$) for the system (5). Each eigenvalue (λ) is then obtained from the equation

$$\bar{S}\bar{\delta} = \lambda \bar{M} \bar{\delta}$$

where $\bar{\delta}$ is the associated eigenvector containing the three displacements for each grid point in the plate.

As an example, the frequencies of vibration and modal patterns are calculated for the square cantilever plate of wedge cross-section shown in Fig. 2. The plan area of the plate is divided into 3×3 and alternatively 5×5 square grid systems, the squares being subdivided into two triangles as shown in mode 1 of Fig. 3.

The frequencies obtained in the present computation are compared with the experimental and calculated results given by Dawe [3] for a rectangular element having a linear thickness variation parallel to one pair of sides only. The frequencies are quoted in Table 4 as dimensionless quantities and the percentage differences between calculated and experimental values are given. For the 5×5 grid system the maximum error in frequency

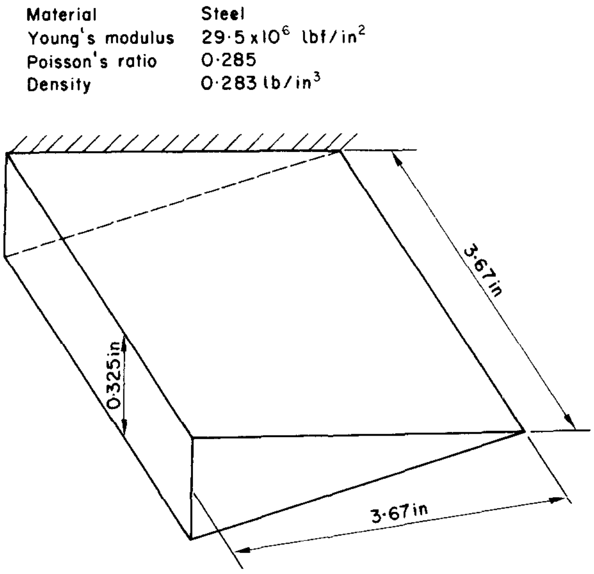

FIG. 2. Square cantilever plate with wedge cross-section.

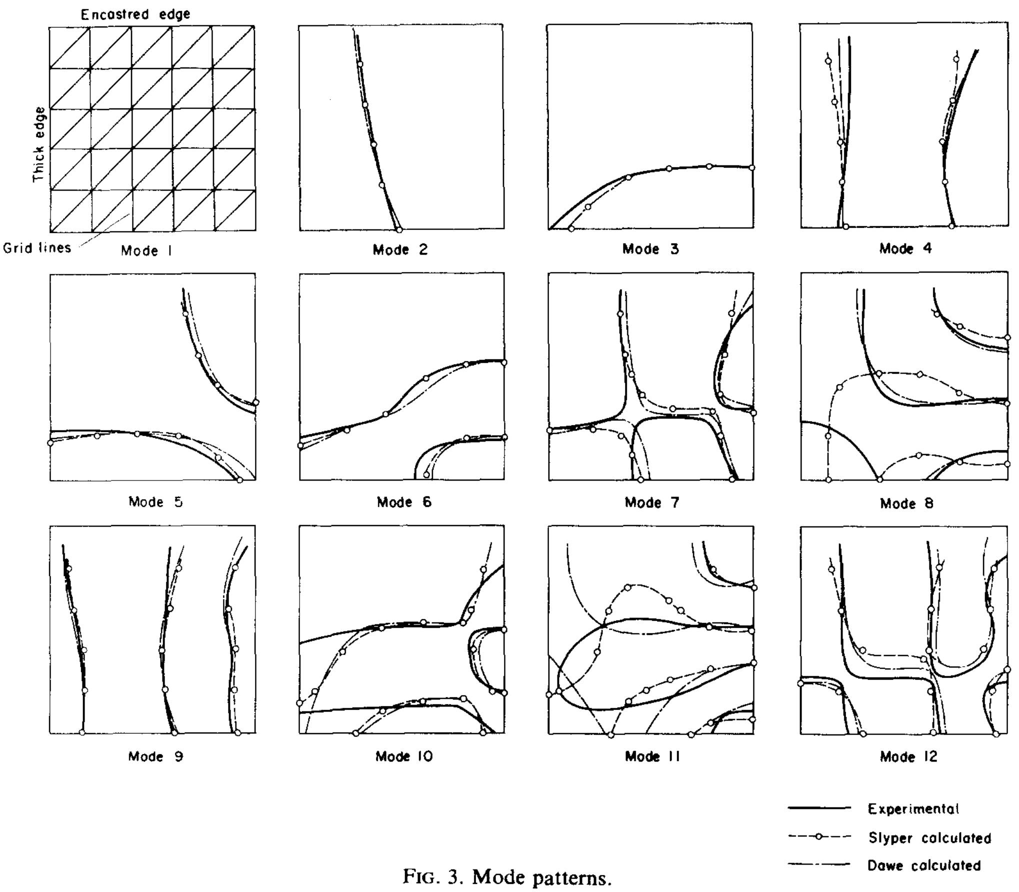

FIG. 3. Mode patterns.

TABLE 3. MASS MATRIX M

$$\frac{\rho\Delta}{15120}$$

$1824t_1+540t_2+540t_3$			
$b_3(177t_1+85.5t_2+49.5t_3)$ $-b_2(177t_1+49.5t_2+85.5t_3)$	$b_3^2(24.5t_1+15.5t_2+6.5t_3)$ $+b_2^2(24.5t_1+6.5t_2+15.5t_3)$ $-2b_2b_3(14.5t_1+7t_2+7t_3)$		
$c_3(177t_1+85.5t_2+49.5t_3)$ $-c_2(177t_1+49.5t_2+85.5t_3)$	$b_3c_3(24.5t_1+15.5t_2+6.5t_3)$ $+b_2c_2(24.5t_1+6.5t_2+15.5t_3)$ $-(b_2c_3+b_3c_2)(14.5t_1+7t_2$ $+7t_3)$	$c_3^2(24.5t_1+15.5t_2+6.5t_3)$ $+c_2^2(24.5t_1+6.5t_2+15.5t_3)$ $-2c_2c_3(14.5t_1+7t_2+7t_3)$	
$432t_1+432t_2+204t_3$	$b_3(76.5t_1+93t_2+34.5t_3)$ $-b_2(40.5t_1+42t_2+31.5t_3)$	$c_3(76.5t_1+93t_2+34.5t_3)$ $-c_2(40.5t_1+42t_2+31.5t_3)$	$540t_1+1824t_2+540t_3$
$b_1(42t_1+40.5t_2+31.5t_3)$ $-b_3(93t_1+76.5t_2+34.5t_3)$	$b_1b_3(6.5t_1+8t_2+5t_3)$ $-b_1b_2(5.5t_1+5.5t_2+5.5t_3)$ $-b_3^2(16t_1+16t_2+5.5t_3)$ $+b_2b_3(8t_1+6.5t_2+5t_3)$	$b_1c_3(6.5t_1+8t_2+5t_3)$ $-b_3c_3(16t_1+16t_2+5.5t_3)$ $-b_1c_2(5.5t_1+5.5t_2+5.5t_3)$ $+b_3c_2(8t_1+6.5t_2+5t_3)$	$b_1(49.5t_1+177t_2+85.5t_3)$ $-b_3(85.5t_1+177t_2+49.5t_3)$
$c_1(42t_1+40.5t_2+31.5t_3)$ $-c_3(93t_1+76.5t_2+34.5t_3)$	$b_3c_1(6.5t_1+8t_2+5t_3)$ $-b_2c_1(5.5t_1+5.5t_2+5.5t_3)$ $-b_3c_3(16t_1+16t_2+5.5t_3)$ $+b_2c_3(8t_1+6.5t_2+5t_3)$	$c_1c_3(6.5t_1+8t_2+5t_3)$ $-c_1c_2(5.5t_1+5.5t_2+5.5t_3)$ $-c_3^2(16t_1+16t_2+5.5t_3)$ $+c_2c_3(8t_1+6.5t_2+5t_3)$	$c_1(49.5t_1+177t_2+85.5t_3)$ $-c_3(85.5t_1+177t_2+49.5t_3)$
$432t_1+204t_2+432t_3$	$b_3(40.5t_1+31.5t_2+42t_3)$ $-b_2(76.5t_1+34.5t_2+93t_3)$	$c_3(40.5t_1+31.5t_2+42t_3)$ $-c_2(76.5t_1+34.5t_2+93t_3)$	$204t_1+432t_2+432t_3$
$b_2(93t_1+34.5t_2+76.5t_3)$ $-b_1(42t_1+31.5t_2+40.5t_3)$	$b_2b_3(8t_1+5t_2+6.5t_3)$ $-b_1b_3(5.5t_1+5.5t_2+5.5t_3)$ $-b_2^2(16t_1+5.5t_2+16t_3)$ $+b_1b_2(6.5t_1+5t_2+8t_3)$	$b_2c_3(8t_1+5t_2+6.5t_3)$ $-b_1c_3(5.5t_1+5.5t_2+5.5t_3)$ $-b_2c_2(16t_1+5.5t_2+16t_3)$ $+b_1c_2(6.5t_1+5t_2+8t_3)$	$b_2(31.5t_1+42t_2+40.5t_3)$ $-b_1(34.5t_1+93t_2+76.5t_3)$
$c_2(93t_1+34.5t_2+76.5t_3)$ $c_1(42t_1+31.5t_2+40.5t_3)$	$b_3c_2(8t_1+5t_2+6.5t_3)$ $-b_2c_2(16t_1+5.5t_2+16t_3)$ $-b_3c_1(5.5t_1+5.5t_2+5.5t_3)$ $+b_2c_1(6.5t_1+5t_2+8t_3)$	$c_2c_3(8t_1+5t_2+6.5t_3)$ $-c_1c_3(5.5t_1+5.5t_2+5.5t_3)$ $-c_2^2(16t_1+5.5t_2+16t_3)$ $+c_1c_2(6.5t_1+5t_2+8t_3)$	$c_2(31.5t_1+42t_2+40.5t_3)$ $-c_1(34.5t_1+93t_2+76.5t_3)$

SYMMETRIC

$b_1^2(6\cdot5t_1+24\cdot5t_2+15\cdot5t_3)$ $+b_3^2(15\cdot5t_1+24\cdot5t_2+6\cdot5t_3)$ $-2b_1b_3(7t_1+14\cdot5t_2+7t_3)$				
$b_1c_1(6\cdot5t_1+24\cdot5t_2+15\cdot5t_3)$ $+b_3c_3(15\cdot5t_1+24\cdot5t_2+6\cdot5t_3)$ $-(b_3c_1+b_1c_3)(7t_1$ $+14\cdot5t_2+7t_3)$	$c_1^2(6\cdot5t_1+24\cdot5t_2+15\cdot5t_3)$ $+c_3^2(15\cdot5t_1+24\cdot5t_2+6\cdot5t_3)$ $-2c_1c_3(7t_1+14\cdot5t_2+7t_3)$			
$b_1(34\cdot5t_1+76\cdot5t_2+93t_3)$ $-b_3(31\cdot5t_1+40\cdot5t_2+42t_3)$	$c_1(34\cdot5t_1+76\cdot5t_2+93t_3)$ $-c_3(31\cdot5t_1+40\cdot5t_2+42t_3)$	$540t_1+540t_2+1824t_3$		
$b_1b_2(5t_1+6\cdot5t_2+8t_3)$ $-b_2b_3(5\cdot5t_1+5\cdot5t_2+5\cdot5t_3)$ $-b_1^2(5\cdot5t_1+16t_2+16t_3)$ $+b_1b_3(5t_1+8t_2+6\cdot5t_3)$	$b_2c_1(5t_1+6\cdot5t_2+8t_3)$ $+b_1c_1(5\cdot5t_1+16t_2+16t_3)$ $-b_2c_3(5\cdot5t_1+5\cdot5t_2+5\cdot5t_3)$ $+b_1c_3(5t_1+8t_2+6\cdot5t_3)$	$b_2(85\cdot5t_1+49\cdot5t_2+177t_3)$ $-b_1(49\cdot5t_1+85\cdot5t_2+177t_3)$	$b_2^2(15\cdot5t_1+6\cdot5t_2+24\cdot5t_3)$ $+b_1^2(6\cdot5t_1+15\cdot5t_2+24\cdot5t_3)$ $-2b_1b_2(7t_1+7t_2+14\cdot5t_3)$	
$b_1c_2(5t_1+6\cdot5t_2+8t_3)$ $-b_3c_2(5\cdot5t_1+5\cdot5t_2+5\cdot5t_3)$ $-b_1c_1(5\cdot5t_1+16t_2+16t_3)$ $+b_3c_1(5t_1+8t_2+6\cdot5t_3)$	$c_1c_2(5t_1+6\cdot5t_2+8t_3)$ $-c_2c_3(5\cdot5t_1+5\cdot5t_2+5\cdot5t_3)$ $-c_1^2(5\cdot5t_1+16t_2+16t_3)$ $+c_1c_3(5t_1+8t_2+6\cdot5t_3)$	$c_2(85\cdot5t_1+49\cdot5t_2+177t_3)$ $-c_1(49\cdot5t_1+85\cdot5t_2+177t_3)$	$b_2c_2(15\cdot5t_1+6\cdot5t_2+24\cdot5t_3)$ $+b_1c_1(6\cdot5t_1+15\cdot5t_2+24\cdot5t_3)$ $-(b_1c_2+b_2c_1)(7t_1+7t_2$ $+14\cdot5t_3)$	$c_2^2(15\cdot5t_1+6\cdot5t_2+24\cdot5t_3)$ $+c_1^2(6\cdot5t_1+15\cdot5t_2+24\cdot5t_3)$ $-2c_1c_2(7t_1+7t_2+14\cdot5t_3)$

TABLE 4. FREQUENCIES OF SQUARE CANTILEVER PLATE OF WEDGE CROSS-SECTION

Mode No.		Dimensionless frequency (α)						Percentage difference				
		Finite element					3 × 3 grid			5 × 5 grid	
	Test	3 × 3 grid			5 × 5 grid						
		Dawe	Slyper	Dawe stepped	Dawe	Slyper	Dawe	Slyper	Dawe stepped	Dawe	Slyper
1	2·302	2·386	2·394	2·385	2·382	2·394	+3·6	+4·3	+3·6	+3·4	+4·3
2	5·566	5·722	5·881	5·469	5·718	5·834	+2·8	+5·7	−1·8	+2·7	+4·8
3	10·69	11·02	11·68	10·21	10·62	10·85	+3·1	+9·3	−3·9	−0·7	+1·5
4	14·51	13·99	15·29	11·50	14·38	15·08	−3·6	+5·4	−20·7	−0·9	+3·9
5	16·25	16·56	16·67	15·42	16·40	16·69	+1·9	+2·6	−5·1	+0·9	+2·7
6	18·38	20·11	20·82	17·58	17·97	19·50	+9·4	+13·3	−4·3	−2·2	+6·1
7	24·98	24·51	26·58	22·22	24·19	25·80	−1·9	+6·4	−11·1	−3·2	+3·3
8	25·59	33·10	32·15	26·60	25·97	27·19	+29·3	+25·7	+3·9	+1·5	+6·2
9	29·28				28·43	30·18				−2·9	+3·1
10	32·31				30·98	33·83				−4·1	+4·7
11	33·26				35·88	36·39				+7·9	+9·4
12	39·40				38·21	41.10				−3·0	+4·3
Size of matrix								36 × 36			90 × 90

$\alpha = p/\sqrt{(D/\rho h L^4)}$
where p = angular frequency
D = flexural rigidity $Eh^3/[12(1-\nu^2)]$
h = maximum plate thickness
L = plate length
ν = Poisson's ratio
E = Young's modulus
ρ = density

for the first ten modes of vibration is 6·2% for the triangular element compared with 4·1% for the rectangle. However, the error range for the triangle is only 4·7% compared with 7·5% for the rectangular element.

The errors in the frequency calculations for the 3×3 grid system are higher than those obtained with the finer mesh. For comparison, the rectangular elements have been replaced by their equivalent of constant thickness [3]. The results for this "stepped" plate are also compared with the new formulation and appear to be particularly poor when calculating the "tram-line" type modes 4 and 7.

In Fig. 3 the calculated modal patterns for the first twelve critical frequencies are compared with those obtained experimentally. Correlation is generally good for the first seven modes of vibration but tends to be poor for modes 8, 10 and 11. Also indicated in the figure are the calculated results for the square element showing that the triangle is as accurate as the rectangular element in addition to being more general in application.

CONCLUSION

The results obtained by the use of this triangular element with linear thickness variation are satisfactory and should be suitable for problems dealing with plates and shells of non-uniform thickness.

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Абстракт—Представляются в явном виде матрицы коэффициентов поперечной жесткости изгиба и массы для трехугольного конечного элемента, обладающего линейным изменением толщины. Этот элемент является более подходящим для общего применении по сравнению с трехугольными и прямоугольными элементами, описанными раньше, для определения статических прогибов и вибрационных характеристик поперечно изгибаемых пластинок.

Применяются матрицы для определения собственных часток вибрации консоли, квадратной в плане, имеющей трехугольное поперечное сечение.